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Stabilization in Viscoelastic Solids *

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Abstract

In this section we study the existence of global solutions for small data and also the asymptotic behaviour of the solution to the full nonlinear problem of a class of viscoelastic equations.

1 Introduction

In this work we study the version abstract of the viscoelastic nonlinear wave equations of memory type. The system in question is the following

$$\partial_t^2 u + M([u])Au - \int_0^t g(t-\tau)N([u])Au \, d\tau = 0, \quad \text{in} \quad L^2(0,T;H)$$
$$u(0) = u_0, \quad \partial_t u(0) = u_1,$$

where by H we are denoting separable Hilbert space and by $[u(t)]$ the nonlinear argument of N and M. Also $M([0]) > 0$.

Such nonlinear model describing a homogeneous and isotropic viscoelastic solid. In recent years have been subject of study of researchers. The particular case $A = -\Delta$ and $[u] = \| A^{1/2}u \|^2$, was studied by Torrejon and Young [14]. The authors showed the existence of global solution, for analytical data and the asymptotic stability when $t \to \infty$. In Nishihara [11] the author consider the wave equation with linear frictional damping and show the existence of global solution for a class of large initial data in $D(A)$ spaces, non analytical but close to an analytical data. Nishihara's result is an important improvement about the question of existence of solution for the nonlinear Kirchhoff equation with weak dissipation, because it provides a large space where the initial data can be taken to produce large existence result.

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2 Notations & Preliminaries

Here $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and the inner product defined over H. By A we are denoting an unbounded nonnegative self-adjoint operator satisfying

$$A\colon D(A)\subset H\to H,$$

and

[V1] The embedding $D(A^r)\hookrightarrow D(A^s)$ is compact for any $r>s\geq 0$. On M and N we impose the following hypotheses

[V2] The functions $M,\ N\colon \mathbb{R}^5\to\mathbb{R}$ are C^2 and $g\colon\mathbb{R}\to\mathbb{R}$ is a C^3-function.

To prove the exponential decay of the solutions we use the following hypothesis on g:

$$0<g(t),\quad -\kappa g(t)\leq g'(t)\leq -cg(t) \tag{2.1}$$

$$|g''(t)|\leq Cg(t), \tag{2.2}$$

$$\alpha=:1-N([0])\int_0^\infty g(\tau)\,d\tau>0 \tag{2.3}$$

to facilitate our computation we introduce the notations

$$\begin{aligned}(g\,\Box\, f)(t) &= \int_0^t g(t-\tau)||f(\tau)-f(t)||^2 d\tau \qquad \text{and}\\ (\eta * v)(t) &= \int_0^t \eta(t-\tau)v(\tau)d\tau.\end{aligned}$$

In this section we study the existence of global solutions for small data and also the asymptotic behaviour of the solution to the full nonlinear problem,

$$\partial_t^2 u+M([u])Au-\int_0^t g(t-\tau)N([u])Au(\tau)d\tau=0, \tag{2.4}$$

$$u(0)=u_0,\qquad \partial_t u(0)=u_1 \tag{2.5}$$

Where $[u]$ is given by

$$[u(t)]=\Big((Au(t),\partial_t u(t))\,,||A^{\frac12}u(t)||^2,||A^{\frac12}\partial_t u(t)||^2,||Au(t)||^2\Big)\in\mathbb{R}^4$$

and M and N satisfies conditions $[V2]$. To explore the dissipative properties of equation (2.4) let us rewrite the equation in the following form,

$$\partial_t^2 u+M([0])Au-N([0])\int_0^t g(t-\tau)Au(\tau)d\tau=P:=R+Q, \tag{2.6}$$

where R and Q are the nonlinear term of equation

$$\begin{aligned}R(t) &= \int_0^t g(t-\tau)\left\{N([u(\tau)])-N(0)\right\}\,Au(\tau)d\tau\\ Q(t) &= \left\{M([u(t)])-M([0])\right\}\,Au(t)\end{aligned}$$

We assume hypotheses (2.1)–(2.2) on the kernel g and instead of the hypotheses (2.3) we use hypotheses:

$$M([0]) - N([0]) \int_0^\infty g(t)\, dt > 0 \tag{2.7}$$

For simplicity and without loss of generality, we suppose that $M(0) = N(0) = 1$, (otherwise we make the change of variables $t \mapsto \sqrt{M([0])}\, t$, and put $\hat{g} := \frac{M([0])}{N([0])} g$) then the hypotheses (2.7) may be written as,

$$1 - \int_0^\infty g(\tau) d\tau = \alpha > 0. \tag{2.8}$$

Rewriting equation (2.6) we have

$$\partial_t^2 u + Au - \int_0^t g(t-\tau) Au(\tau) d\tau = P := R + Q, \tag{2.9}$$

Here the energy $E(t,v)$ of the solution to the problem (2.4)-(2.5) is defined by

$$E(t,v) = \frac{1}{2}\left\{ \|\partial_t v\|^2 + \left(1 - \int_0^t g(\tau)\, d\tau\right) \|A^{1/2} v\|^2 + g \,\square\, A^{1/2} v \right\}. \tag{2.10}$$

3 Main result

Theorem 3.1 *Let us that hypotheses [V1] and [V2] holds and let us take g satisfying (2.1)-(2.3). Consider $\epsilon > 0$ such that the initial data*

$$(u_0, u_1) \in D(A^{3/2}) \times D(A)$$

satisfies

$$\left\| A^{\frac{3}{2}} u_0 \right\|^2 + ||Au_1||^2 < \epsilon.$$

Then, there exist only one solution u of equation (2.4), such that

$$u \in C^2([0,\infty[, D(A^{1/2})) \cap C^1([0,\infty[, D(A)) \cap C([0,\infty[, D(A^{3/2})).$$

In addition we have that the energy $E(t,u)$ defined in (2.10) satisfies

$$E(t) \le E(0)\, e^{-\gamma t} \quad \text{for all} \quad t \ge 0 \quad \text{and} \quad \gamma > 0.$$

Proof.- Applying the operator A to equation (2.9) and using Remark 3.2 (see Gómez et al. [6]) for $v = Au$ we have

$$\frac{d}{dt}\mathcal{L}_\nu(t, Au) \le -\kappa \mathcal{L}_\nu(t, Au) - \frac{\nu}{2}\left\{ g(t)\|A^{3/2}u\|^2 + g \,\square\, A^{3/2} u \right\} + \tag{3.1}$$

$$+\nu\,(AP, A\partial_t u) + \left(AP, A\partial_t w + \frac{g(0)}{2} Aw \right)$$

Since M and N are continuous functions, for all $\delta > 0$, there exist $\epsilon > 0$, such that

$$|\sigma|_{\mathbb{R}^4} < c_2 \epsilon \Rightarrow |M(\sigma) - M([0])| < \delta \qquad \text{and} \qquad |N(\sigma) - N([0])| < \delta,$$

From the local existence, Theorem 2.1 (see Gómez et al. [6]), there exist $0 < T_0 \leq T_{\max}$, such that

$$\mathcal{M}(t, Au) := \left\| A^{\frac{3}{2}} u(t) \right\|^2 + ||A\partial_t u||^2 + g \,\square\, A^{3/2}u \leq d\epsilon \quad \text{em} \quad [0,\ T_0[.$$

where $d \geq 1$ to be fixed later. Let us consider

$$T^* = \sup \left\{ T_1^* > 0 \colon E(t) \leq d\epsilon \quad \text{in} \quad [0,\ T_1^*[\right\}$$

We have two cases: (i) $T^* = T_{\max}$, (ii) $T^* < T_{\max}$. The first one implies that the solutions u is bounded so, we have $T_{\max} = \infty$. So, we only consider case (ii) Suppose that $T^* < T_{\max}$ and $T_{\max} < \infty$ then we have

$$\begin{aligned} &|[u(t)]|_{\mathbb{R}^4} < c_2\epsilon \quad \Rightarrow \quad |M([u]) - M([0])| < \delta \\ &|N([u]) - N([0])| < \delta \quad \text{in} \quad [0,\ T^*[. \end{aligned} \tag{3.2}$$

Denoting by α_1 the expression,

$$\alpha_1 = \max_{|s| \leq c_2\epsilon} \{\partial_{x_i} M(\mathbf{x}) \quad : i = 1, 2, 3, 4, 5\}.$$

From, hypotheses of global existence of solutions of the nonlinear viscoelastic system, Remark 4.1 (see Gómez et al. [6]) we have that

$$\begin{aligned} \left| \frac{d}{dt} \{M([0]) - M([u])\} \right| &\leq 2\alpha_1 \left\{ ||A^{1/2}\partial_t u(t)|| \right\} \\ &+ 2\alpha_1 \left\{ ||A^{3/2}u(t)|| + ||A\partial_t u(t)|| \right\} \leq c_3\epsilon, \end{aligned} \tag{3.3}$$

Note that

$$\begin{aligned} (AQ, A\partial_t u) &= \{M([u]) - M([0])\} \left(A^2 u, A\partial_t u \right) \\ &= -\frac{1}{2} \left(\frac{d}{dt} \{M([u]) - M([0])\} \right) \left\| A^{3/2} u \right\|^2 \\ &\quad + \frac{1}{2} \frac{d}{dt} \left(\{M([u]) - M([0])\} \left\| A^{3/2} u \right\|^2 \right) \\ &\leq c_3\delta \left\| A^{3/2} u \right\|^2 + \frac{1}{2} \frac{d}{dt} \left(\{M([u]) - M([0])\} \left\| A^{3/2} u \right\|^2 \right) \end{aligned}$$

$$\begin{aligned} (AR, A\partial_t u) &= \frac{d}{dt} (AR, Au) - (A\partial_t R, Au) \\ &= \frac{d}{dt} (AR, Au) - g(0) \{N([u(t)]) - N([0])\} \left(A^2 u, Au \right) \\ &+ \int_0^t g'(t-\tau) \{N([u(\tau)]) - N([0])\} A^2 u(\tau)\, d\tau Au \\ &\leq \frac{d}{dt} (AR, Au) + \delta g(0) \|A^{3/2}u\|^2 + \delta C \left\{ g \,\square\, A^{3/2}u + \left\| A^{3/2} u \right\|^2 \right\}. \end{aligned}$$

From where it follows that

$$\begin{aligned}(AP, A\partial_t u) \quad &\leq \quad c_3\epsilon \left\{ \left\| A^{3/2}u \right\|^2 + g \,\square\, A^{3/2}u \right\} \\ &\quad + \frac{d}{dt}\left\{ \frac{1}{2}\left\{ M([u]) - M([0]) \right\} \left\| A^{3/2}u \right\|^2 + (AR, Au) \right\}\end{aligned}$$

Similarly we have

$$\begin{aligned}\left(AP, A\partial_t w + \frac{1}{2}Aw \right) &\leq c_3\delta \left\{ \left\| A^{3/2}u \right\|^2 + g \,\square\, A^{3/2}u \right\} \\ &\quad + \frac{1}{2}\frac{d}{dt}\left(\left\{ M([u]) - M([0]) \right\} \left\| A^{3/2}u \right\|^2 \right) \\ &\quad + \frac{d}{dt}(AR, Au)\,.\end{aligned}$$

Denoting by

$$\mathcal{S}(t) = 2\left\{ M([u]) - M([0]) \right\} \left\| A^{3/2}u \right\|^2 + 2\,(AR, Au)\,.$$

From (3.1) and taking ϵ and δ small enough it follow that

$$\begin{aligned}\frac{d}{dt}\left\{ \mathcal{L}(t) - \mathcal{S}(t) \right\} \quad &\leq \quad -\frac{\kappa}{2}\mathcal{L}(t); \\ |\mathcal{S}(t)| \quad &< \quad \frac{c_0}{2}\delta\, \mathcal{M}(t).\end{aligned}$$

Recalling the definition of $\mathcal{S}$ we have

$$\frac{1}{2}\mathcal{M}(t) \leq \mathcal{L}(t) - \mathcal{S}(t) \leq 2\mathcal{M}(t), \tag{3.4}$$

and

$$\frac{d}{dt}\left\{ \mathcal{L}(t) - \mathcal{S}(t) \right\} \leq -\frac{\kappa}{2}\{\mathcal{L}(t) - \mathcal{S}(t)\},$$

which implies that

$$\mathcal{L}(t) - \mathcal{S}(t)) \leq \{\mathcal{L}(0) - \mathcal{S}(0)\}e^{-\gamma t},$$

where $\gamma = \dfrac{\kappa}{2}$. From above inequality together with (3.4) we have

$$\mathcal{M}(t) \leq \frac{2}{c_0}\{\mathcal{L}(0) - \mathcal{S}(0)\}e^{-\gamma t} \leq \frac{4c_1}{c_0}\mathcal{M}(0)e^{-\gamma t} \quad \text{for all} \quad t \in [0, T^*[$$

The next step is to show that $T_{max} = \infty$. To do it we reason by contradiction. Let us suppose that $T^* < T_{max} < \infty$ and that $T^* = T_1^*$. Thus we have

$$\mathcal{M}(t) \leq d\mathcal{M}(0)\, e^{-\gamma t} < d\,\epsilon\, e^{-\gamma t}. \tag{3.5}$$

Letting $t \to T^* = T_1^*$, it follows that

$$\mathcal{M}(T_1^*) \leq d\, e^{-\gamma T_1^*}\, \epsilon < d\,\epsilon,$$

which is a contradiction to the maximality of T_1^*. Hence, $T_{\max} = \infty$ so, the solution is global in time. From where our conclusion follows □

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